Problem 1. Potential games (6 points)

Consider a game between two players. Each player has two pure strategies A and B. Both players are **maximizers**. The payoff matrix is given as:

	A	B
A	$(3,3)$	$(0,5)$
B	$(5,0)$	$(2,2)$

In each entry, the first element denotes the utility of the row player and the second element denotes the utility of the column player. The above is a potential game.

a) For the function $\Phi : \{A,B\} \times \{A,B\} \to \mathbb{R}$ below, complete the two missing values $x_1, x_2 \in \mathbb{R}$ so that Φ will be a potential function for the game. (1 point)

$$\Phi = \begin{matrix} & A & B \\ A & 0 & x_1 \\ B & 2 & x_2 \end{matrix}$$

b) For a given game with N players, and player i's costs $J_i : \Gamma_1 \times \cdots \times \Gamma_N \to \mathbb{R}$ for $i = 1, \ldots, N$, show that if Φ and Ψ are both potential functions, then there exists a constant $c \in \mathbb{R}$ such that $\Phi(s) - \Psi(s) = c$, for every joint strategy $s \in \Gamma_1 \times \cdots \times \Gamma_N$. (2 points)

c) Consider a student social network where each student communicates to their friends, and this connection is represented by a graph with N vertices and E edges. Namely, if student i and j are friends, then there is an edge between vertices i and j. On a given topic (such as if the course ME-429 is good or not), each student has an opinion, captured by $x_i \in \{0,1\}$, $i = 1, \ldots, N$, where $0, 1$ represent disagree and agree, respectively. Each student feels some conformity to their friends and thus, tries to adjust her opinion by **minimizing** deviation of her opinion from those of her friends: $J_i(x) = \sum_{j \in \mathcal{N}_i} |x_i - x_j|$, where $\mathcal{N}_i$ denotes the set of friends of student i, and $x \in \{0,1\}^N$.

   i. Show that the game characterized by $\{J_i\}_{i=1}^N$ is a potential game. Hint: verify $\frac{1}{2}\sum_{i=1}^N J_i(x)$ is an exact potential function for the game. (1 point)
   ii. Determine two Nash equilibria of the game above. (1 point)
   iii. Show that the best response of a student to the neighbors' strategies $\{x_j\}_{j \in \mathcal{N}_i}$ is the majority opinion of her neighbors. (1 point)

Problem 2. Zero-Sum game (7 points)

Consider a zero-sum game between two players captured by matrix A below. The minimizer chooses a row and the maximizer chooses a column.

$$A = \begin{bmatrix} 3 & 2 & 3 \\ 2 & 0 & -2 \\ -3 & 2 & 0 \end{bmatrix}.$$

a) Identify any rows or columns that are strictly dominated. Iteratively eliminate strictly dominated rows and columns. (2 points)

b) Consider now a zero-sum game with utilities captured by the matrix $B \in \mathbb{R}^{2\times 2}$ below. Consider the strategies $y^* = \left[\frac{1}{3}, \frac{2}{3}\right]$ for the minimizer (row player) and $z^* = \left[\frac{1}{6}, \frac{5}{6}\right]$ for the maximizer (column player). Verify that (y^*, z^*) is a saddle point equilibrium. (2 points)

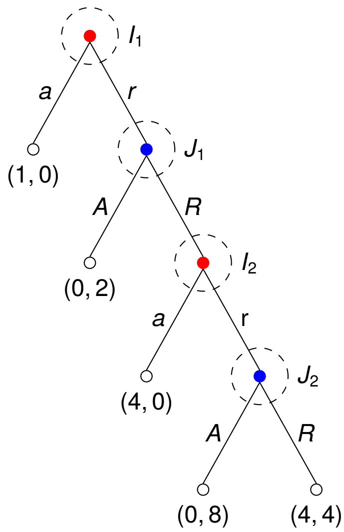
$$B = \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix}$$

c) Consider a 2-player zero-sum game, played sequentially and under the full information feedback. Let Alice be the minimizer and Bob be the maximizer. The cost for Alice is captured by a matrix $C \in \mathbb{R}^{m\times n}$.

   i. Consider Alice playing first. Show that Alice's strategy arising from backward induction corresponds to her security strategy. (1 point)
   ii. Based on the above and your knowledge of the min-max inequality, argue why Alice would prefer to go second. (1 point)
   iii. Under which condition Alice would be indifferent to going first or second? (1 point)

Problem 3. Backward induction in extensive form games (7 points)

Consider the multi-stage game shown in the figure on the right. There is a pot of money, starting with 1 CHF. At each iteration, the decision-maker (player 1 or player 2) can decide to accept the pot ending up with the outcome shown in the left leaf of the tree, or to refuse the pot, at which point the pot size doubles and the next player makes the same decision.



Game tree
(red: player 1, blue: player 2)

a) Is this a feedback game? Is this a full information game? (1 point)

b) Given the information sets shown, how many strategies does each player have? (1 point)

c) Use backward induction to compute the subgame perfect Nash equilibrium strategy and the outcome of the game. Show your work by drawing the game tree on your solution sheet and showing the backward induction. (2.5 points)

d) Consider the social welfare function of maximizing the sum of both players' payoffs $C : \Gamma_1 \times \Gamma_2 \to \mathbb{R}$. What would be the players' strategies maximizing the social welfare and the corresponding social welfare value? (2 points)

e) The price of anarchy is defined as $\frac{C(s^{\text{opt}})}{\min_{s \in S^{\text{NE}}} C(s)}$, where $s^{opt} \in \Gamma_1 \times \Gamma_2$ is a socially optimal strategy and the minimum in the denominator is taken over Nash equilibrium strategies, S^{NE}. What is the price of anarchy in this game? (0.5 point)